Problem Set 3

Due: February 13th, 5pm

1 Formal proofs

For each problem in this section, you must give a complete and rigorous proof. Your arguments should be clear, logically ordered, and written in full sentences. In particular, state all assumptions explicitly and define all notation. Justify every nontrivial step.

1. (1 point) Let s, t be positive integers such that $s > t$ and s, t are relatively prime (i.e. $\gcd(s, t) = 1$). Show that

$$st, \quad \frac{s^2 - t^2}{2}, \quad \text{and} \quad \frac{s^2 + t^2}{2}$$

are pairwise relatively prime (i.e. the gcd of any two of them is 1).

2. (1 point) Prove that $\sqrt{3}$ is irrational.

3. (1 point) Prove that $\log_{10}(5)$ is irrational

4. (2 points) Suppose that x, y, z are positive integers such that $\gcd(x, y, z) = 1$ and satisfy

$$x^2 + 2y^2 = z^2.$$

   (a) Prove that x, z have the same parity (i.e. are both either even or odd). *Hint: Rearrange into an equation* $2y^2 = z^2 - x^2$.

   (b) Prove that y is even.

2 Demonstrations

For problems in this section, you still need to give complete mathematical reasoning to support your answers. I should be able to follow and understand your work, but it doesn't have to be organized into a formal proof.

5. (3 points) Let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \colon a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$, and consider the function

$$N \colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z} \qquad N(a + b\sqrt{-5}) = a^2 + 5b^2$$

   (a) Show that if $x, y \in \mathbb{Z}[\sqrt{-5}]$ then both $x + y$ and $x \cdot y$ are in $\mathbb{Z}[\sqrt{-5}]$.

   (b) Show that N satisfies the following properties:

      (i) $N(x) \geq 0$ for all $x \in \mathbb{Z}[\sqrt{-5}]$ and $N(x) = 0$ if and only if $x = 0$.

      (ii) $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{Z}[\sqrt{-5}]$.

   (c) An element $x \in \mathbb{Z}[\sqrt{-5}]$ is a called a *unit* if there exists y such that $x \cdot y = 1$. Show that x is a unit if and only if $N(x) = 1$, and use this to find all units in $\mathbb{Z}[\sqrt{-5}]$.

   (d) An element $p \in \mathbb{Z}[\sqrt{-5}]$ is called *prime* if whenever $p = x \cdot y$ for some $x, y \in \mathbb{Z}[\sqrt{-5}]$, either x or y is a unit.

   (e) Determine which of the elements $2, 3, 5, 7, 1 + \sqrt{-5}, 2 - 3\sqrt{-5}$ are prime.

   (f) Show that 6 does not have a unique factorization into primes. (Part of this exercise is for you to formulate what unique prime factorization means in $\mathbb{Z}[\sqrt{-5}]$!)

3 Explorations

In this section, I'm looking for answers that are supported by evidence, but you don't have to prove your answers.

6. (1 point) Consider the equation $6x + 15y + 27z = d$.

   (a) What are all of the integer solutions when $d = 12$?

   (b) Make a conjecture to answer the following question: For which $d \in \mathbb{Z}$ does $6x + 15y + 27z = d$ have a solution (in the integers)?

7. (1 point) Let $p > 3$ be a prime integer. When does p remain prime as an element of $\mathbb{Z}[\sqrt{-5}]$? Compile a list of primes up to 100, and determine if they remain prime in $\mathbb{Z}[\sqrt{-5}]$. Do you have a conjecture for which p remain prime? *Hint: Talk to me if you need help getting started, and I strongly recommend using SageMath to do this.*

8. (Bonus) *This question is optional, it will not be graded and is here for fun ☺.* Generalized the above exercise to $\mathbb{Z}[\sqrt{d}]$ for $d \in \mathbb{Z}$ squarefree.