Problem Set 2

Due: February 6th, 5pm

For each problem, you must give a complete and rigorous proof. Your arguments should be clear, logically ordered, and written in full sentences.

In particular, state all assumptions explicitly and define all notation. Justify every nontrivial step.

1. Prove that the product of two odd integers is odd.

2. Prove that if n^2 is divisible by 3, then n is divisible by 3.

3. Prove or disprove the following statements.

   (a) $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R} : x^2 > y$

   (b) $\exists y \in \mathbb{R}\, \forall x \in \mathbb{R} : x^2 > y$.

   (c) (*Hint: You may use results that we have proven in class.*)

   $$\forall a \in \mathbb{Z}, \forall n \in \mathbb{Z}_{>0} : (\gcd(a,n) = 1 \implies (\exists x \in \mathbb{Z} \colon n \mid 1 - ax))\,.$$

4. Prove that for all integer $n \geq 1$, the following formula holds

   $$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

5. Prove that every integer $n > 1$ is either prime or can be written as a product of prime numbers. *Hint: Use strong induction.*

6. Let n be an odd integer. We say that n is Type 1 (respectively Type 3) if n can be written as $4k+1$ (respectively $4k+3$) for some integer k.

   (a) Let n, m be odd integers. Consider the statement "nm is a Type 3 integer only if *exactly one* of n, m is a Type 3 integer". Write this as an "If... then..." statement, and give a direct proof, using only the definition.

   (b) Prove by induction the statement that if $x_1, \ldots, x_n$ are odd integers such that the product $\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdots x_n$ is a Type 3 integer, then *at least* one of $x_1, x_2, \ldots, x_n$ is a Type 3 integer.

(c) Prove that there are infinitely many prime numbers that are Type 3. *Hint: Modify Euclid's proof that there are infinitely many primes by considering numbers* $4p_1 \cdots p_n + 3$.

7. Let i be an integer, and let P_i be the statement "There are infinitely prime numbers of the form $3k + i$ for some integer k".

   (a) Prove that $P_i \implies P_{i+3}$.

   (b) Write down the statement $P_1 \vee P_2$ (where $\vee$ is the logical or) as a sentence, and prove it. In your proof, you may use that there are infinitely many prime numbers.